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Nonlinear systems which have finite-dimensional
 \mathcal{H}_∞ suboptimal central controllers

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Abstract

Following up the work of Başar and Bernhard [2], we have recently derived in [12] the nonlinear central controller solving the nonlinear (standard) \mathcal{H}_∞ suboptimal control problem. This nonlinear central controller is an *infinite-dimensional* system, and resembles very much the solution in nonlinear stochastic filtering or nonlinear deterministic filtering. After showing that in the *linear* case the nonlinear central controller reduces to the finite-dimensional central controller as obtained in [4], we consider in the present note the question if there are truly nonlinear systems having finite-dimensional central controllers. Guided by similar considerations in nonlinear stochastic and deterministic filtering, see especially [5], we characterize a specific class of nonlinear systems having finite-dimensional central controllers. This class can be regarded as the deterministic \mathcal{H}_∞ analogue of the class of nonlinear systems admitting finite dimensional filters as identified by Benes [3].

We consider nonlinear control systems of the form

$$\begin{aligned}\dot{x} &= a(x) + b(x)u + g(x)d_1 \\ y &= c(x) + d_2 \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}\end{aligned}\quad (1)$$

where $x = (x_1, \dots, x_n)$ are local coordinates for a smooth state space manifold M . Furthermore, $u \in \mathbb{R}^m$ denote the *control inputs*, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^r$ the *exogenous inputs* (disturbances and/or references), $y \in \mathbb{R}^p$ the *measured outputs*, and $z \in \mathbb{R}^s$ the *to-be-controlled outputs* (tracking errors, cost variables). The maps $a(x), b(x), g(x), c(x), h(x)$ are all assumed to be C^k , with $k \geq 2$. Throughout we assume the existence of a fixed equilibrium x_0 , i.e. $a(x_0) = 0$, and without loss of generality we set $x_0 = 0$, and also we let $c(0) = h(0) = 0$. Now let γ be a fixed positive constant. The \mathcal{H}_∞ suboptimal control problem (for disturbance attenuation level γ) is to find a compensator

$$\begin{aligned}\dot{\xi} &= k(\xi, y) \\ u &= m(\xi, y)\end{aligned}\quad (2)$$

where $\xi = (\xi_1, \dots, \xi_\nu)$ are local coordinates for a manifold M_c (the state space of the compensator), with $k(0, 0) = 0$ and $m(0, 0) = 0$, such that the closed-loop system (1), (2) has L_2 -gain *less than or equal to* γ , in the sense that there exists a nonnegative constant K depending on $x(0), \xi(0)$ and zero for $x(0) = 0, \xi(0) = 0$, such that

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 + K \quad (3)$$

for all $d(\cdot)$ and $T \geq 0$, with $z(\cdot)$ denoting the closed-loop response for initial condition $x(0), \xi(0)$. If some *observability* conditions are satisfied (with regard to the outputs z !), then property (3) will also imply internal stability of the closed-loop system. For further motivation and details we refer e.g. to [1], [7], [9], [10], [12].

The state of the art for this problem is, very roughly, as follows. The state feedback problem (i.e. $y = x$) is reasonably well-understood [9], [10], [7]. For the full (dynamic output feedback) problem appealing necessary conditions have been found generalizing the famous necessary and sufficient conditions for linear systems obtained in [4], see [1], [11], [12]. Much effort and ingenuity has been put in obtaining various sufficient conditions, see e.g. [1], [6], [7], but the problem is still largely open.

Following up the work of Başar and Bernhard [2], and using older work on nonlinear deterministic filtering [8], [5], we have recently taken another approach to the full \mathcal{H}_∞ suboptimal control problem. In fact, in [12] we have shown that under suitable technical conditions the \mathcal{H}_∞ suboptimal control problem is solved by the controller

$$\begin{aligned}\dot{\hat{x}} &= \left[a(\hat{x}) - b(\hat{x})b^T(\hat{x})P_x^T(\hat{x}) + \frac{1}{\gamma^2}g(\hat{x})g^T(\hat{x})P_x^T(\hat{x}) \right] \\ &\quad + \gamma^2 [S_{xx}(\hat{x}, t)]^{-1} \frac{\partial c^T}{\partial x}(\hat{x}) [y(t) - c(\hat{x})] \\ u &= -b^T(\hat{x})P_x^T(\hat{x})\end{aligned}\quad (4)$$

with $P \geq 0$ being the minimal solution to the Hamilton-Jacobi equation

$$\begin{aligned}P_x(x)a(x) + \frac{1}{2}P_x(x) \left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x) \right] P_x^T(x) \\ + \frac{1}{2}h^T(x)h(x) = 0, \quad P(0) = 0,\end{aligned}\quad (5)$$

(implying the solvability of the *state feedback* \mathcal{H}_∞ suboptimal control problem, see [9], [10]), and with R being a solution to the non-stationary Hamilton-Jacobi equation

$$\begin{aligned}R_t(x, t) + R_x(x, t)a(x) + \frac{1}{2}\frac{1}{\gamma^2}R_x(x, t)g(x)g^T(x)R_x^T(x, t) \\ + \frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2 c^T(x)c(x) + \gamma^2 c^T(x)y(t) \\ + R_x(x, t)b(x)u(t) - \frac{1}{2}\gamma^2 \|y(t)\|^2 + \frac{1}{2}\|u(t)\|^2 = 0\end{aligned}\quad (6)$$

(with $u(t)$ given as the output of (4)), such that $S(x, t) := R(x, t) - P(x)$ has a unique minimum $\hat{x}(t)$ for every t with invertible Hessian matrix $S_{xx}(\hat{x}(t), t)$.

This controller is obtained as the solution of a certain min-max optimization problem with imperfect state measurements, see [2]; and thus is called the nonlinear *central controller*, since in the linear case this optimization problem is known [2] to yield the central controller of [4]. In fact, in the sequel we will *directly* demonstrate how in the linear case the nonlinear central controller reduces to the central controller of [4].

Although the first part of the nonlinear central controller, i.e. (4), has an appealing "worst-case disturbance" observer structure (see [2], [1], [12]), in general the gain matrix $[S_{xx}(\hat{x}, t)]^{-1}$ cannot be computed off-line, since the partial differential equation (6) for $R(x, t)$ is directly driven by the measured outputs $y(t)$, as well as by $u(t)$. Thus the nonlinear central controller is *infinite-dimensional*.

Now suppose (1) is linear, i.e.

$$\begin{aligned}\dot{x} &= Ax + Bu + Gd_1 \\ y &= Cx + d_2 \\ z &= \begin{bmatrix} Hx \\ u \end{bmatrix}\end{aligned}\quad (7)$$

It is readily checked (see [5] for the same argument in the context of deterministic filtering) that the solution $R(x, t)$ of (6) in this case takes the form $R(x, t) = c(t) + \ell^T(t)x + \frac{1}{2}x^T Q(t)x$ with $R_{xx}(x, t) = Q(t) \geq 0$ satisfying

$$\dot{Q}(t) + A^T Q(t) + Q(t)A + \frac{1}{\gamma^2}Q(t)GG^T Q(t) + H^T H - \gamma^2 C^T C = 0 \quad (8)$$

It follows that $R_{xx}(x, t) = Q(t)$ can be computed off-line, and in fact can be taken to be the maximal constant matrix $Q > 0$ solving the algebraic Riccati equation

$$A^T Q + Q A + \frac{1}{\gamma^2} G G^T Q + H^T H - \gamma^2 C^T C = 0, \quad (9)$$

(Notice that the dual Riccati equation (FARE) of [4] is obtained from (9) by dividing by γ^2 and pre- and post-multiplication by $\gamma^2 Q^{-1}$). Furthermore the stationary Hamilton-Jacobi equation (5) reduces to an algebraic Riccati equation, and it is immediately seen that the resulting linear controller (4) is precisely the central controller of [4].

A logical question is now: are there any other systems, apart from the linear ones, for which $R_{xx}(x, t)$ and thus $S_{xx}(x, t)$ can be computed off-line, and therefore the nonlinear central controller reduces to the finite-dimensional controller (4)? In order to study this problem we take the same approach as used in [5] for the deterministic nonlinear filtering problem, i.e. we consider the Hamiltonian function $H(x, p)$ corresponding to the Hamilton-Jacobi equation (6):

$$\begin{aligned} H(x, p) = & p^T a(x) + \frac{1}{2\gamma^2} p^T g(x) g^T(x) p + \frac{1}{2} h^T(x) h(x) \\ & - \frac{1}{2} \gamma^2 c^T(x) c(x) + \gamma^2 c^T(x) y(t) \\ & + p^T b(x) u(t) - \frac{1}{2} \gamma^2 \|y(t)\|^2 + \frac{1}{2} \|u(t)\|^2, \end{aligned} \quad (10)$$

with $p \in \mathbb{R}^n$ denoting the co-state. (Notice that (6) can be rewritten as $\bar{R}_t + H(x, \bar{R}_x^T) = 0$.) It is natural (see [5]) to consider the following type of canonical transformations $(x, p) \mapsto (x, \bar{p})$, where

$$p = \bar{p} + V_x^T(x) \quad (11)$$

for some C^k function $V(x)$. Then the Hamiltonian $H(x, p)$ transforms into $\bar{H}(x, \bar{p}) := H(x, \bar{p} + V_x^T(x))$, leading to the transformed Hamilton-Jacobi equation $\bar{R}_t + \bar{H}(x, \bar{R}_x^T) = 0$ given as

$$\begin{aligned} \bar{R}_t(x, t) + \bar{R}_x(x, t) \left[a(x) + \frac{1}{\gamma^2} g(x) g^T(x) V_x^T(x) \right] \\ + \frac{1}{2\gamma^2} \bar{R}_x(x, t) g(x) g^T(x) \bar{R}_x^T(x, t) + \frac{1}{2} h^T(x) h(x) \\ - \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2\gamma^2} V_x(x) g(x) g^T(x) V_x^T(x) + V_x(x) a(x) \\ + \gamma^2 c^T(x) y(t) + \bar{R}_x(x, t) b(x) u(t) + V_x(x) b(x) y(t) \\ - \frac{1}{2} \gamma^2 \|y(t)\|^2 + \frac{1}{2} \|u(t)\|^2 = 0 \end{aligned} \quad (12)$$

It immediately follows that \bar{R} is a solution of (12) if and only if $\bar{R} + V$ is a solution of (6). Now assume that

$$\begin{aligned} b(x) \text{ and } g(x) \text{ are constant,} \\ c(x) \text{ is linear in } x, \end{aligned} \quad (13)$$

and suppose that the nonlinear system (1) has the property that $V(x)$ can be found such that

$$\begin{aligned} a(x) + \frac{1}{\gamma^2} g(x) g^T(x) V_x^T(x) \text{ is at most linear in } x, \\ \frac{1}{2} h^T(x) h(x) - \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2\gamma^2} V_x(x) g(x) g^T(x) V_x^T(x) \\ + V_x(x) a(x) \text{ is at most quadratic in } x, \\ V_x(x) b(x) \text{ is at most linear in } x. \end{aligned} \quad (14)$$

Then, as in the linear case, it follows that the solution $\bar{R}(x, t)$ of (12) can be written as $\bar{R}(x, t) = \bar{c}(t) + \bar{\ell}(t)x + \frac{1}{2} x^T \bar{Q}(t)x$, where $\bar{Q}(t) = \bar{R}_{xx}(x, t)$ is the solution of a differential Riccati equation of the same type as (8), and thus can be computed off-line (without knowing $y(t)$ and $u(t)$). Therefore also the Hessian $R_{xx}(x, t)$ of the solution $R(x, t) = \bar{R}(x, t) + V(x)$ of (6) can be computed off-line, and thus in this case the nonlinear central controller reduces to the finite-dimensional controller (4)! Summarizing:

Theorem Consider the nonlinear system (1). Suppose the \mathcal{H}_∞ sub-optimal control problem is solvable by the nonlinear central controller (4), (6). Assume that $b(x)$, $g(x)$ and $c(x)$ are as in (13), while $V(x)$

can be found such that (14) holds. Then there exists a solution $R(x, t)$ of (6) for which $R_{xx}(x, t)$ can be computed off-line, and the nonlinear central controller reduces to the finite-dimensional controller (4).

Example Consider the almost linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u + d_1, & y &= x_2 + d_2 \\ \dot{x}_2 &= -x_2 & z &= \begin{bmatrix} x_2^3 \\ u \end{bmatrix} \end{aligned}$$

Then $V(x_1, x_2) = \frac{1}{12} x_2^6$ satisfies (12), and thus the nonlinear central controller reduces to a finite dimensional controller. \square

It would be nice to characterize the class of nonlinear systems covered by the above theorem in a more explicit and coordinate free way. Notice however that (as in nonlinear filtering, see [3], [5]) it is expected to be only a small subset of all nonlinear systems. On the other hand, we recall from [12] that even if the central controller is inherently infinite-dimensional, then still there may exist finite dimensional controllers also solving the (sub-)optimal \mathcal{H}_∞ control problem.

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